MA 335, Spring 2012

Assignment 7.

This homework is due *Thursday* March 8.

There are total 42 points in this assignment. 38 points is considered 100%. If you go over 38 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

(1) (a) [2pt] Show that the sets $\{d;\ d>0 \text{ and } d \text{ divides } n\}$ and $\{\frac{n}{d};\ d>0 \text{ and } d \text{ divides } n\}$ are equal. Deduce that

$$\sum_{d|n} d = \sum_{d|n} \frac{n}{d} \quad \text{and} \quad \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

(If you see $\prod$ notation for the first time, it's like $\sum$ notation, only for product. For example, $\prod_{i=1}^{4} = 1\cdot 2\cdot 3\cdot 4$.)

(b) [3pt] Prove that

$$\prod_{d|n} d = n^{\tau(n)/2}.$$

(*Hint:* Write $\prod_{d|n} d \cdot \prod_{d|n} d = \prod_{d|n} d \cdot \prod_{d|n} \frac{n}{d}$.)

Comment. It would seem that we may get an irrational number after we take a root on the right hand side. However, it cannot happen because the left hand side is always integer. Problem 5a gives another, more immediate explanation why the right hand side cannot be non-integer.

(2) [3pt] (6.1.1) Let m, n be positive integers and $p_1,\dots,p_r$ be the distinct primes that divide at least one of m or n. Then m and n may be written as

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \quad k_i \ge 0,$$
$$m = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}, \quad j_i \ge 0.$$

Prove that

$$\gcd(m,n) = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}, \quad \operatorname{lcm}(m,n) = p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r},$$

where $u_i = \min\{k_i, j_i\}$ and $v_i = \max\{k_i, j_i\}$. ($\operatorname{lcm}(a,b,\dots,c)$ is the *least common multiple* of numbers $a,b,\dots,c$, i.e. the least positive integer that is a multiple of each of numbers $a,b,\dots,c$.)

(3) (a) [2pt] (6.1.3) Deduce from Problem 2 that $\gcd(m,n)\cdot\operatorname{lcm}(m,n) = mn$ for all positive integers m,n.

(b) [2pt] Is it true that $\gcd(m,n,k)\cdot\operatorname{lcm}(m,n,k) = mnk$ for all positive integers m,n,k?

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(4) [3pt] (6.1.6) For any integer $n \geq 1$, establish that $\tau(n) \leq 2\sqrt{n}$. (*Hint:* In each pair s.t. $d_1 d_2 = n$, at least one of numbers $\leq \sqrt{n}$.)

(5) (6.1.7) Prove the following.
  (a) [3pt] $\tau(n)$ is an odd integer if and only if n is a perfect square.
  (b) [3pt] $\sigma(n)$ is an odd integer if and only if n is of the form m^2 or $2m^2$. (*Hint:* If p is an odd prime, then $1 + p + \ldots + p^k$ is odd if and only if k is even.)

(6) (6.1.14) For $k \geq 2$, show each of the following:
  (a) [2pt] $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$.
  (b) [2pt] If $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$.
  (c) [2pt] If $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies the equation $\sigma(n) = 2n + 2$.

  Comment. It is an open question if there are any positive integers such that $\sigma(n) = 2n + 1$.

(7) [4pt] Numbers with the property $\sigma(n) = 2n$ are called *perfect numbers*. In other words, perfect numbers are those equal to sum of their divisors, excluding the number itself. For example, $28 = 1 + 2 + 4 + 7 + 14$. In problem 6b we proved that numbers of the form $n = 2^{k-1}(2^k - 1)$, with $2^k - 1$ prime, are perfect. Prove the partial converse: if a perfect number is *even*, then n has the form $n = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime. (*Hint:* Represent n as $2^k \cdot m$, where m is odd. Use multiplicativity of σ.)

  Comment. The statement above says nothing about odd perfect numbers, and there is a good reason: it is an open question whether they exist.

(8) ($\sim$6.1.22) τ and σ are particular cases of a family of number theoretic functions σ_s, $s \in \mathbb{R}$:

$$\sigma_s(n) = \sum_{d|n} d^s,$$

so $\tau = \sigma_0$ and $\sigma = \sigma_1$.
  (a) [3pt] (6.1.8) Prove that $\sigma_{-1} = \sigma(n)/n$. (*Hint:* Multiply both sides by n.)
  (b) [2pt] (6.1.17) Show that for every fixed $s \in \mathbb{R}$, function n^s is multiplicative.
  (c) [3pt] Prove that σ_s is a multiplicative function for every $s \in \mathbb{R}$. (*Hint:* Use item 8b.)
  (d) [3pt] If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_4}$ is the prime factorization of n, then

$$\sigma_s(n) = \left(\frac{p_1^{s(k_1+1)} - 1}{p_1^s - 1} \right) \left(\frac{p_2^{s(k_2+1)} - 1}{p_2^s - 1} \right) \cdots \left(\frac{p_r^{s(k_r+1)} - 1}{p_r^s - 1} \right).$$

  (*Hint:* Compute $\sigma_s(n)$ in the case $n = p^k$. Then use multiplicativity.)